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Heat transfer in a radial liquid jet

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The heat transfer in a radial liquid jet is investigated. In the region where a similarity solution of the momentum equation is available solutions of the energy equation describing the effects of viscous dissipation, initial heating and wall heating are obtained in closed form. Two examples illustrating the work are discussed. In the second of these an approximate method, based on the heat flux equation, is used to describe the initial development of the thermal boundary layer.

1. Introduction

A radial liquid jet is formed when a smooth jet of liquid falls vertically on to a horizontal plane and spreads out radially over it as, for example, water falling from a tap to the bottom of an empty sink. The liquid spreads out in a thin layer until the depth increases suddenly forming a hydraulic jump. In order to discuss the motion of the fluid in the thin layer before the hydraulic jump, the assumptions of boundary-layer theory are applied which require that the Reynolds number of the impinging jet should be large. An important contribution to the theory of radial liquid jets has been made by Watson (1964). He found a similarity solution of the boundary-layer equations governing such flow and also considered by approximate methods the initial growth of the boundary layer from the stagnation point where the similarity solution does not hold. Later it was investigated by Riley (1962*a*) in his study of radial jets with swirl.

In studying the velocity distribution Watson found it convenient to divide the flow into four different regions which pass continuously into one another.

(i) The region near the central stagnation point where the radial distance $r = O(a_0)$, a_0 being the radius of the impinging jet. Here the boundary-layer thickness is $O(\nu a_0/U_0)^{\frac{1}{2}}$ where U_0 is the speed of the impinging jet and ν the kinematic viscosity.

(ii) When $r \gg a_0$ conditions in region (i) are unimportant and the boundary layer grows like the Blasius boundary layer on a flat plate.

(iii) As r increases the viscous stresses affect more and more fluid across the jet and the boundary layer increases in thickness until it absorbs the whole layer of fluid. The velocity profile then gradually changes from Blasius-type to the similarity profile mentioned earlier.

(iv) At large distances from the stagnation point the way in which the flow originated becomes unimportant and the final similarity form is attained.

The hydraulic jump associated with this type of flow will ultimately terminate the region of flow under consideration.

The problem of the distribution of temperature in a radial liquid jet is studied here by conforming to a similar division of the flow. Section 2 contains the appropriate equations of motion and the similarity solution of the momentum equation first found by Watson is briefly discussed. In § 3 similarity solutions of the energy equation appropriate to region (iv) are obtained for a wide variety of temperature conditions.

In the first of the two examples described in § 4, part of the wall is assumed to be thermally insulated, the rest being maintained at a constant temperature different from that of the initial jet. A solution of the energy equation is found for this latter part which is chosen to correspond to region (iv) described above. In the second example the whole wall is maintained at a constant temperature so that both the temperature and velocity distributions have to be studied in all the four regions described earlier. Regions (i) and (iii) are neglected following Watson (1964) and Riley (1962*b*). In region (ii) an approximate method using integrated forms of the boundary-layer equations and polynomials of the fourth degree for the temperature and velocity functions is employed. The neglect of regions (i) and (iii) and Watson's approximate method in region (ii) are discussed at the end of § 2. For this second example, which may have important practical applications, expressions are given for the Nusselt number in regions (ii) and (iv) for several values of the Prandtl number. Illustrations showing the effectiveness of radial liquid jets for cooling purposes are also included.

In the examples discussed above we assume that the contribution to the temperature of the liquid in the jet due to viscous heating is negligible compared to the applied heating. Throughout this work the boundary-layer equations are assumed to be appropriate, temperature differences are taken to be small, and μ, the viscosity and ρ, the density are assumed to be constant.

2. Equations of motion

Using boundary-layer approximations the momentum, continuity and energy equations governing the incompressible laminar flow of a liquid jet striking a plane horizontal wall at right angles and spreading out radially over it are respectively

$$u(\partial u/\partial r)+w(\partial u/\partial z) = \nu(\partial^2 u/\partial z^2), \tag{2.1}$$

$$\partial(ru)/\partial r+\partial(rw)/\partial z = 0, \tag{2.2}$$

and

$$u(\partial T/\partial r)+w(\partial T/\partial z) = (\nu/\sigma)\,\partial^2 T/\partial z^2+(\nu/S)\,(\partial u/\partial z)^2. \tag{2.3}$$

Here r, z are distances measured along the wall from the jet axis and normal to it respectively; u, w are the corresponding velocity components and ν, T, σ, S denote respectively the kinematic viscosity, temperature, Prandtl number and specific heat of the liquid in the jet. The boundary conditions are

$$u = w = 0 \quad \text{at} \quad z = 0, \tag{2.4}$$

$$\partial u/\partial z = 0 \quad \text{at} \quad z = \phi(r), \tag{2.5}$$

and

$$\partial T/\partial z = 0 \quad \text{at} \quad z = \phi(r). \tag{2.6}$$

Other boundary conditions on T necessary to specify the problem completely will be introduced later. Conditions (2.5) and (2.6) express the fact that the shearing stress and heat transfer are zero at the free surface $z = \phi(r)$. With the boundary conditions (2.4) and (2.5) Watson (1964) has shown that a similarity solution of the momentum equation is available. This may be written as

$$\psi = (3\sqrt{(3)}\, Q/\pi) f(\eta), \quad \eta = \{3\sqrt{(3)}\, Qr/\pi\nu(r^3 + l^3)\}\, z, \tag{2.7}$$

where ψ is the stream function defined as

$$ur = \partial\psi/\partial z, \quad wr = -\partial\psi/\partial r, \tag{2.8}$$

and l is an arbitrary constant length which depends on the initial development of the boundary layer. Watson estimates $l = 0{\cdot}567 a_0 R^{\frac{1}{3}}$, where R is the jet Reynolds number $2\pi Q/\nu a_0$. The constant Q is given by the condition of constant volume flux per radian, namely

$$\int_0^{\phi(r)} ur\, dz = Q. \tag{2.9}$$

The function f satisfies the ordinary differential equation

$$\left.\begin{aligned} &f''' + 3f'^2 = 0, \\ \text{with} \qquad &f(0) = f'(0) = f''(1) = 0, \end{aligned}\right\} \tag{2.10}$$

the free surface having been chosen to be $\eta = 1$. It is convenient, for what follows, to make the transformation

$$f = \tfrac{1}{2}\alpha_1^2 f_1, \tag{2.11}$$

where the constant α_1 is chosen so that

$$f_1'(1) = 1. \tag{2.12}$$

Thus the equation satisfied by f_1 is

$$f_1''' + \tfrac{3}{2}\alpha_1^2 f_1'^2 = 0, \tag{2.13}$$

with the same boundary conditions as for f in (2.10). Integrating equation (2.13) once

$$f_1'' = \alpha_1(1 - f_1'^3)^{\frac{1}{2}}, \tag{2.14}$$

and so

$$\alpha_1 \eta = \int_0^{f_1'} (1 - s^3)^{-\frac{1}{2}}\, ds, \tag{2.15}$$

which with (2.12) gives

$$\alpha_1 = \int_0^1 (1 - s^3)^{-\frac{1}{2}}\, ds = 1{\cdot}402. \tag{2.16}$$

Also, from (2.14)

$$\int_0^1 f_1'(\eta)\, d\eta = \frac{1}{\alpha_1}\int_0^1 f_1'(1 - f_1'^3)^{-\frac{1}{2}}\, df_1' = \frac{2\pi}{3\sqrt{(3)}\alpha_1^2}. \tag{2.17}$$

The velocity function f' is displayed graphically in figure 1.

If $r \gg a_0$, conditions prevailing in region (i) where $r = O(a_0)$ are not important and in the approximate analysis discussed below, and in § 4, region (i) is ignored. For his approximate solution, in region (ii), Watson used the Kármán–Pohlhausen method with

$$u = U_0 f_1'(\eta), \quad \eta = z/\delta(r), \tag{2.18}$$

where $f_1'(\eta)$ is the similarity profile defined by (2.15) and $\delta(r)$ is the boundary-layer thickness. This technique has the effect of suppressing region (iii) in which the velocity profile changes to its final similarity form. In fact, Riley (1962*b*) has shown that in region (iii), when the boundary layer fills the whole of the moving layer of fluid, any disturbance to the similarity velocity profile is $O(r^{-45})$. Thus the final similarity form is attained very rapidly. Substitution of the approximate velocity profile (2.18) in the momentum integral equation for radial flow

$$\left(\frac{d}{dr}+\frac{1}{r}\right)\int_0^\delta u(U_0-u)\,dz = \nu\left(\frac{\partial u}{\partial z}\right)_{z=0}, \tag{2.19}$$

gives, on integration

$$r^2\delta = \frac{\sqrt{(3)}\,\alpha_1^3}{(\pi-\sqrt{(3)}\,\alpha_1)}\frac{\nu r^3}{U_0}+C, \tag{2.20}$$

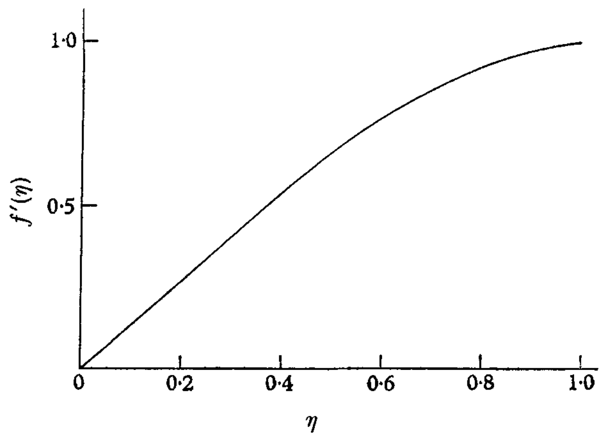

FIGURE 1. The velocity function $f'(\eta)$ from (2.10).

where C is a constant. A consideration of the order of magnitude shows that $C = O(a_0^3/r^3)$ relative to the other terms there and hence can be neglected when $r \gg a_0$. Thus when $r_0 > r \gg a_0$

$$\delta^2 = \frac{\sqrt{(3)}\,\alpha_1^3}{(\pi-\sqrt{(3)}\,\alpha_1)}\frac{\nu r a_0^2}{2Q}, \tag{2.21}$$

where r_0 is the station at which the boundary layer just absorbs the whole flow. Watson calculated the value of r_0 from the condition that the volume flux through the boundary layer reaches the value Q there, and thus obtained on the basis of the above approximate solution

$$r_0 = 0{\cdot}3155 a_0 R^{\frac{1}{3}}. \tag{2.22}$$

3. Similarity solutions of the energy equation

In region (iv) where the solution of (2.1) and (2.2) is given by (2.7) and (2.8), the energy equation (2.3) may be written, with (r, η) as independent variables, as

$$\frac{\partial^2 T}{\partial \eta^2}-\frac{\sigma(r^3+l^3)}{r^2}f'\frac{\partial T}{\partial r} = -\frac{27^2Q^4\sigma}{\pi^4\nu^2 S(r^3+l^3)^2}f''^2, \tag{3.1}$$

with the boundary condition

$$\partial T/\partial\eta = 0 \quad \text{at} \quad \eta = 1. \tag{3.2}$$

At the wall either the wall temperature T_w or heat transfer, proportional to $(\partial T/\partial\eta)_{\eta=0}$, may be prescribed. In the examples discussed in § 4 we shall restrict ourselves to the case of constant wall temperature or, if the wall is thermally insulated, zero heat transfer.

A particular integral of (3.1) can be found in the form

$$T = T_0 + C_0(r^3+l^3)^{-2}\theta_0(\eta), \tag{3.3}$$

where T_0 and C_0 are constants. Equation (3.1) also has complementary functions of the form

$$T = C_1 + C_2(r^3+l^3)^{-\alpha}\theta_n(\eta), \tag{3.4}$$

with θ_n satisfying

$$\theta_n'' + 3\alpha\sigma f'\theta_n = 0. \tag{3.5}$$

A complete picture of the temperature distribution prevailing in a particular problem can thus be obtained by adding to the particular integral (3.3) appropriate complementary functions of the type (3.4).

The effects of viscous heating, wall heating and initial heating will now be studied separately.

(*a*) *Viscous heating*

To study the effects of viscous dissipation on the temperature distribution in the jet we require a particular integral of (3.1) in the form (3.3) where, with $C_0 = -27^2Q^4/\pi^4\nu^2S$, $\theta_0(\eta)$ satisfies

$$\theta_0'' + 6\sigma f'\theta_0 = \sigma f''^2, \tag{3.6}$$

with boundary conditions

$$\left.\begin{array}{lll} \text{either} & \theta_0(0) = 0 & \text{if the wall is maintained at constant temperature,} \\ \text{or} & \theta_0'(0) = 0 & \text{for a thermally insulated wall,} \\ \text{and} & \theta_0'(1) = 0 & \text{from (3.2).} \end{array}\right\} \tag{3.7}$$

It is convenient to change the independent variable in (3.6) and (3.7) from η to a new variable t with

$$t = f_1'^3. \tag{3.8}$$

Thus, from (2.11), (3.6) and (3.8) we have, as the equation for θ_0

$$t(1-t)\,d^2\theta_0/dt^2 + (\tfrac{2}{3}-\tfrac{7}{6}t)\,d\theta_0/dt + \tfrac{1}{3}\sigma\theta_0 = \tfrac{1}{36}\sigma\alpha_1^4 t^{-\frac{1}{3}}(1-t). \tag{3.9}$$

The boundary conditions (3.7) now become

$$\left.\begin{array}{ll} & \theta_0(0) = 0, \quad \text{or} \quad [t^{\frac{2}{3}}\,d\theta_0/dt]_{t=0} = 0, \\ \text{and} & [(1-t)^{\frac{1}{2}}\,d\theta_0/dt]_{t=1} = 0. \end{array}\right\} \tag{3.10}$$

A solution of equation (3.9) may be obtained in terms of a generalized hypergeometric function. Thus in the usual notation,

$$\theta_0 = \tfrac{1}{8}\sigma\alpha_1^4 t^{\frac{2}{3}} + \tfrac{3}{160}\sigma\alpha_1^4(1-\sigma)\,t^{\frac{5}{3}}\,{}_3F_2\begin{pmatrix} a, b, c; t \\ d, e; \end{pmatrix}, \tag{3.11}$$

where $12a = [21+(1+48\sigma)^{\frac{1}{2}}]$, $12b = [21-(1+48\sigma)^{\frac{1}{2}}]$, $c = 1$, $d = \frac{7}{3}$ and $e = \frac{8}{3}$. From this particular integral and the complementary functions of (3.9) (θ_{21}, θ_{22} of § 3 (*c*) below in which we set $\alpha = 2$) solutions may be constructed which satisfy either of the conditions (3.10) at $t = 0$, together with that at $t = 1$. For $\sigma = 1$

the solution (3.11) gives $\theta_0 = \frac{1}{2}f'^2$ as indicated by the quadratic term in the well-known Crocco relation

$$T + u^2/2S = A + Bu, \tag{3.12}$$

where A and B are constants.

We expect the effects of viscous dissipation in a liquid jet to be small and indeed, in what follows, we shall assume that it can be neglected compared with the applied heating.

(b) *Wall heating*

When the wall is maintained at a constant temperature T_2, we need a complementary function of the form (3.4) with $\alpha = 0$. Thus

$$T = T_1 + (T_2 - T_1)\,\theta_1(\eta),$$

where T_1 is the temperature of the incident jet. From (3.5) $\theta_1(\eta)$ now satisfies $\theta_1'' = 0$, which, with the boundary conditions $\theta_1(0) = 1$, $\theta_1'(1) = 0$ has the trivial solution $\theta_1 = 1$. This solution reflects the physical situation that ultimately all the fluid is raised to temperature T_2. The manner in which the fluid attains this constant temperature depends upon the initial heating of the fluid, the effects of which we now consider.

(c) *Initial heating*

We now require further complementary functions of the type (3.4). Thus when viscous dissipation effects are negligible we may write the temperature as

$$T = T_2 + C_2(r^3 + l^3)^{-\alpha}\,\theta_2(\eta),$$

where $\theta_2(\eta)$ satisfies the differential equation (3.5).

We may note that if the liquid from which the jet is formed is, as we shall assume, at a uniform temperature T_1 and the wall over which the fluid flows is thermally insulated then the fluid remains at constant temperature as no heat is transferred to or from the fluid across either the wall or free surface.

When the wall is maintained at constant temperature T_2 the appropriate boundary conditions for θ_2 are the first and third of those in (3.7). The transformation from η to the new variable t defined in (3.8) reduces (3.5) to the hypergeometric equation

$$t(1-t)\,d^2\theta_2/dt^2 + (\tfrac{2}{3} - \tfrac{7}{6}t)\,d\theta_2/dt + \tfrac{1}{6}\sigma\alpha\theta_2 = 0, \tag{3.13}$$

the boundary conditions now being the first and third of those in (3.10). The determination of α in equation (3.13) is an eigenvalue problem. In view of the form of the boundary condition at $t = 1$ it is convenient to choose the following as the solutions of (3.13)

$$\theta_{21} = F(p, q; \tfrac{1}{2}; 1-t),$$

and

$$\theta_{22} = (1-t)^{\frac{1}{2}}\,F(\tfrac{2}{3} - p, \tfrac{2}{3} - q; \tfrac{3}{2}; 1-t),$$

where p and q are given from

$$p + q = \tfrac{1}{6}, \quad pq = -\tfrac{1}{6}\alpha\sigma. \tag{3.14}$$

The boundary condition at the free surface determines θ_{21} as the required solution and that at the wall requires that

$$[\theta_{21}]_{t=0} = (-\tfrac{1}{2})!\,(-\tfrac{2}{3})!/(-p-\tfrac{1}{2})!\,(-q-\tfrac{1}{2})! = 0;$$

with (3.15) this gives p, q and α as

$$p = (h+\tfrac{1}{2}), \quad q = -(h+\tfrac{1}{3}),$$
$$\alpha = (1+2h)(1+3h)/\sigma \quad \text{where} \quad h = 0, 1, 2, \ldots.$$

Therefore, using a standard transformation of the hypergeometric function, we have

$$\theta_2 = A_h t^{\frac{1}{3}} F(-h, h+\tfrac{5}{6}; \tfrac{1}{2}; 1-t), \tag{3.15}$$

where A_h are constants. Thus, writing $\lambda(r) = (r^3+l^3)/(r_1^3+l^3)$ and including other constants in A_h we have

$$T = T_2 + t^{\frac{1}{3}} \sum_{h=0}^{\infty} A_h \lambda^{-(1+2h)(1+3h)/\sigma} F(-h, \tfrac{5}{6}+h; \tfrac{1}{2}; 1-t). \tag{3.16}$$

For $\sigma = 1$ the leading term in (3.16) is given by the linear term in u in the Crocco relation (3.12). The hypergeometric function in (3.16) is a polynomial and is related to the Jacobi polynomials $P_h^{(i,j)}(\xi)$ by

$$P_h^{(i,j)}(\xi) = [(i+h)!/i!\,h!]\,F(-h, h+i+j+1; i+1; \tfrac{1}{2}-\tfrac{1}{2}\xi).$$

The Jacobi polynomials are orthogonal in $-1 < \xi < 1$ with the weight factor $(1-\xi)^i(1+\xi)^j$. Therefore, if the temperature distribution is known at any station $r = r_1$ the constants A_h can be calculated from

$$\frac{A_h}{(T_1-T_2)} = \frac{(2h+\frac{5}{6})(h-\frac{1}{2})!\,(h-\frac{1}{6})!}{\pi h!\,(h+\frac{1}{3})!} \int_0^1 \left(\frac{T-T_2}{T_1-T_2}\right)_{r=r_1} (1-t)^{-\frac{1}{2}} F(-h, \tfrac{5}{6}+h; \tfrac{1}{2}; 1-t)\,dt. \tag{3.17}$$

The heat transfer across the wall, per unit area, is given by

$$Q_w = -\left(k\frac{\partial T}{\partial z}\right)_{z=0} = -\frac{3\sqrt{(3)}\,(-\frac{1}{2})!\,(-\frac{2}{3})!\,kQ\alpha_1 r}{\pi\nu(r_1^3+l^3)} \sum_{h=0}^{\infty} \frac{A_h \lambda^{-[(1+2h)(1+3h)/\sigma]-1}}{(h-\frac{1}{2})!\,(-h-\frac{4}{3})!}, \tag{3.18}$$

where k is the thermal conductivity.

4. Examples

As a first example we consider the case where part of the wall, $r < r_1$, is assumed to be thermally insulated and the rest maintained at a constant temperature T_2. The end-point, $r = r_1$, of the thermally insulated part is assumed to be in region (iv) as described in the introduction. Therefore, in the regions (i), (ii) and (iii) and in particular at $r = r_1$ the temperature is uniform everywhere and equal to T_1. Thus the constants A_h occurring in (3.16) may be determined from (3.17), with $[(T-T_2)/(T_1-T_2)]_{r=r_1} = 1$, as

$$\frac{A_h}{(T_1-T_2)} = \frac{(-1)^h\sqrt{(3)}\,(-\frac{1}{3})!\,(h-\frac{1}{6})!\,(2h+\frac{5}{6})}{2\pi^{\frac{3}{2}} h!\,(h+\frac{1}{2})\,(h+\frac{1}{3})}. \tag{4.1}$$

The first six values of A_h are given in table 1 where a value $\sigma = 5$ appropriate to water, has been chosen for the Prandtl number. The local rate of heat transfer across the wall for $r > r_1$ is displayed graphically in figure 2 where, for convenience, we have taken $r_1 = r_0$.

As a second example we consider the case when the wall is maintained at a constant temperature throughout; thus the temperature distribution in the liquid passes smoothly through the stages (i)–(iii) before attaining the similarity form in region (iv). We have already explained, in § 2, why regions (i) and (iii) may be ignored and we have also discussed there the method employed by

h	0	1	2	3	4	5
$A_h/(T_1-T_2)$	$+1\cdot187$	$-0\cdot280$	$+0\cdot150$	$-0\cdot100$	$+0\cdot074$	$-0\cdot059$

TABLE 1. Values of the constants A_h calculated from (4.1)

Watson for an approximate solution in region (ii). To determine the temperature distribution in region (ii) we use an approximate method in which both the momentum integral equation and the heat flux equation—an integrated form of the energy equation—are used. To simplify the calculation polynomials of the fourth degree are assumed for the velocity and temperature functions. The heat flux equation, obtained by integrating the energy equation across the boundary layer and neglecting frictional heat, is

$$\frac{d}{dr}\left\{\int_0^\infty ru(T-T_1)\,dz\right\} = -\frac{\nu r}{\sigma}\left(\frac{\partial T}{\partial z}\right)_{z=0}. \tag{4.2}$$

The velocity and temperature distributions are assumed to have the forms

$$u = U_0(2\eta-2\eta^3+\eta^4), \tag{4.3}$$

$$T-T_1 = (T_2-T_1)\,(1-2\eta_T+2\eta_T^3-\eta_T^4), \tag{4.4}$$

where $\eta = z/\delta$ and $\eta_T = z/\delta_T$, $\delta(r)$ and $\delta_T(r)$ being the velocity and thermal boundary-layer thicknesses respectively. The ratio δ_T/δ will be denoted by Δ. The form of the temperature distribution assumed in (4.4) ensures identical velocity and temperature profiles required for the case $\sigma = \Delta = 1$ in the absence of frictional heating. Inserting (4.3) and (4.4) in (4.2) we obtain

$$U_0\Delta^2H(\Delta)\,d(r\delta)^2/dr = 4\nu r^2/\sigma, \tag{4.5}$$

where, for $\Delta < 1$ (which will be the case when the jet is formed from water)

$$H(\Delta) = \tfrac{2}{15}\Delta-\tfrac{3}{140}\Delta^3+\tfrac{1}{180}\Delta^4.$$

With δ known (4.5) is an equation for Δ. To determine δ in this case we substitute (4.3) into the momentum integral equation (2.19) to get

$$\delta^2 = \tfrac{420}{37}\nu r/U_0, \tag{4.6}$$

where, as before, the constant of integration may be set equal to zero. Equation (4.6) is analogous to the result (2.21) obtained by Watson. Equation (4.5) with (4.6) now gives

$$\Delta^2H(\Delta) = \tfrac{37}{315}\sigma^{-1}. \tag{4.7}$$

For the reasons given at the end of § 2 region (iii) can be ignored. The approximate solution of the energy equation for region (ii) may be matched with the solution (3.16) at the station $r = r_0$ where the flow attains its similarity form.

Thus, in this case, $r_1 = r_0$ and the quantity r_0 is determined by the condition that the volume flux through the boundary layer attains the value Q there. Thus (2.9) with (4.3) and (4.6) gives

$$r_0 = 0{\cdot}243 a_0 R^{\frac{1}{3}}. \tag{4.8}$$

To estimate l we apply the condition that the jet thickness is continuous at $r = r_0$. Thus

$$\left[\frac{420}{37}\frac{\nu r_0}{U_0}\right]^{\frac{1}{2}} = \frac{\pi\nu(r_0^3+l^3)}{3\sqrt{(3)}\,Qr_0}, \tag{4.9}$$

which, with (4.8) and remembering that $U_0 a_0^2 = 2Q$, gives

$$l = 0{\cdot}558 a_0 R^{\frac{1}{3}}, \tag{4.10}$$

which may be compared with Watson's value $0{\cdot}567 a_0 R^{\frac{1}{3}}$.

h	0	1	2	3	4
$A_h/(T_1-T_2)$	$+1{\cdot}165$	$-0{\cdot}220$	$+0{\cdot}072$	$-0{\cdot}021$	$+0{\cdot}001$

TABLE 2. Values of the constants A_h obtained numerically from (3.17)

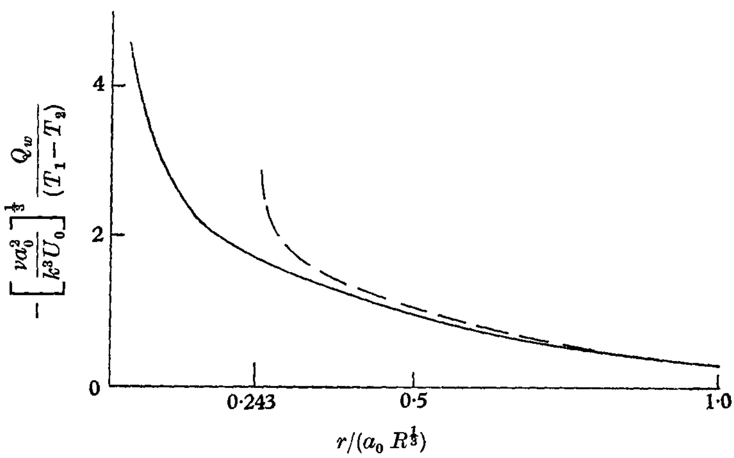

FIGURE 2. The heat transfer across the wall. Example 1, – – – – –; example 2, ———.

The constants A_h in this case are given by (3.17) where $[(T-T_2)/(T_1-T_2)]_{r=r}$ is evaluated from (4.4). The values of A_h, found by numerical integration with $\sigma = 5$, $\Delta = 0{\cdot}570$, are shown in table 2.

The local rate of heat transfer across the wall in this case is also displayed graphically in figure 2 where it is compared with the previous example. For $r \leqslant r_0$ it is calculated from the approximate solution described above and for $r > r_0$ from equation (3.18).

Since the cooling of hot surfaces by thin liquid layers, of the type envisaged in this paper, may have important practical applications we include for this second example expressions for the Nusselt number in regions (ii) and (iv) for several values of the Prandtl number σ. The Nusselt number Nu is defined as

$$\mathrm{Nu} = \bar{Q}_w/ka_0(T_1-T_2), \tag{4.11}$$

where $\bar{Q}_w$ is the heat transfer across a disk of radius r centre the jet axis, thus

$$\bar{Q}_w = \int_0^r 2\pi r Q_w \, dr. \tag{4.12}$$

Then, for $r/a_0 R^{\frac{1}{3}} \leqslant 0{\cdot}243$, from (4.4), (4.6), (4.7) and (4.11) we have

$$\mathrm{Nu}/R = -\Delta^*(r^3/a_0^3 R)^{\frac{1}{3}}, \tag{4.13}$$

whilst for $r/a_0 R^{\frac{1}{3}} > 0{\cdot}243$ from (3.17), (3.18), (4.11) and (4.13) we have the Nusselt number, based on the first five terms of the relevant series, given as

$$\frac{\mathrm{Nu}}{R} = \sum_{h=0}^{4} D_h \lambda^{\gamma_h} - D. \tag{4.14}$$

σ	Δ^*	D_0	D_1	D_2	D_3	D_4	D
1	1·404	0·850	0·019	$0{\cdot}000_1$	$-0{\cdot}000_2$	$0{\cdot}000_3$	1·038
2	1·800	1·753	0·054	0·004	$0{\cdot}000_1$	$0{\cdot}000_1$	2·027
3	2·080	2·673	0·099	0·013	0·001	−0·001	3·035
4	2·282	3·587	0·144	0·022	0·003	$-0{\cdot}000_3$	4·030
5	2·463	4·502	0·189	0·033	0·007	$0{\cdot}000_2$	5·027
6	2·624	5·418	0·236	0·045	0·010	0·001	6·025
7	2·752	6·330	0·280	0·056	0·014	0·002	7·012
8	2·894	7·243	0·324	0·066	0·017	0·003	8·001
9	3·019	8·155	0·368	0·076	0·021	0·004	8·987
10	3·119	9·065	0·411	0·086	0·023	0·004	9·965

TABLE 3. The constants Δ^*, D_h and D as defined in (4.15)

In the expressions (4.13) and (4.14) the constants $\Delta^*(\sigma)$, $D_h(\sigma)$, $\gamma_h(\sigma)$ and $D(\sigma)$ are defined as

$$\left.\begin{aligned} &\Delta^* = 1{\cdot}404\Delta^{-1}, \quad D_h = \surd(3)\,(-\tfrac{1}{2})!\,(-\tfrac{4}{3})!\,\alpha_1 A_h/\pi(T_1-T_2)\,(h-\tfrac{1}{2})!\,(-h-\tfrac{4}{3})!\,\gamma_h,\\ &\gamma_h = -(1+2h)\,(1+3h)/\sigma, \quad D = 0{\cdot}120\Delta^* + \sum_{h=0}^{4} D_h, \end{aligned}\right\} \tag{4.15}$$

where Δ is the solution of (4.7). The quantities Δ^*, D_h $(h = 0, 1, 2, 3, 4)$ and D are given for $\sigma = 1(1)\,10$ in table 3. We may also note that in this case $\lambda(r)$, from its definition in § 3 and equations (4.8), (4.10) may be written as

$$\lambda = 5{\cdot}319\, r^3/a_0^3 R + 0{\cdot}926.$$

As an illustration of the above results the Nusselt number is displayed, for three specific values of σ, in figure 3 showing the effectiveness of this type of flow for cooling purposes.

The approximate method described here is inferior to that of Watson's, described in § 2, since the assumed quartic profile (4.3) does not join on smoothly with the similarity solution at $r = r_0$. However, as indicated earlier, the transition region (iii) in which the velocity profile attains its final similarity form will be small and it is sufficient for our purposes, especially in view of the enormous simplifications in the analysis, to assume the quartic profiles (4.3) and (4.4).

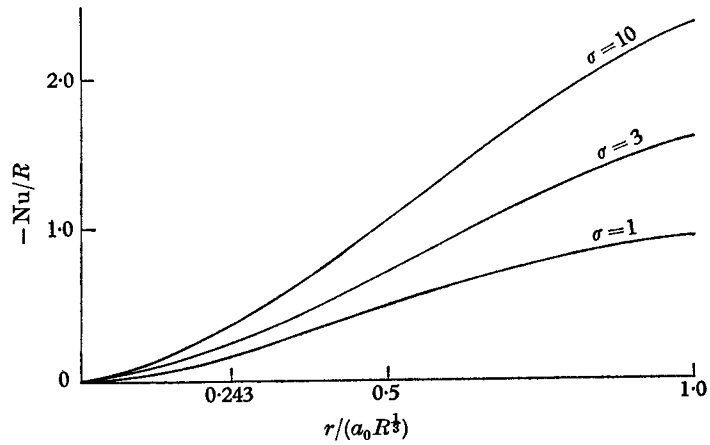

FIGURE 3. The Nusselt number, for $\sigma = 1, 3$ and 10, from (4.13) and (4.14) using the results in table 3.

The thin layer of fluid in which we have been investigating the temperature distribution is terminated by a sudden increase in depth at a station $r = r_2$, say. This is a hydraulic jump and an estimate of r_2 has been made by Watson (1964). Watson also extends his analysis to the case of turbulent flow which is outside the scope of the present work.

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